Martin's maximum revisited

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Abstract

We present several results relating the general theory of the stationary tower forcing developed by Woodin with forcing axioms. In particular we show that, in combination with strong large cardinals, the forcing axiom MM^{++} makes the Π_2-fragment of the theory of $H_{\aleph_2}$ invariant with respect to stationary set preserving forcings that preserve BMM. We argue that this is a close to optimal generalization to $H_{\aleph_2}$ of Woodin's absoluteness results for $L(\mathbb{R})$. In due course of proving this we shall give a new proof of some of Woodin's results.

1 Introduction

In this introduction we shall take a long detour to motivate the results we want to present and to show how they stem out of Woodin's work on Ω-logic. We tried to make this introduction comprehensible to any person acquainted with the theory of forcing as presented for example in [7]. The reader may refer to subsection 1.1 for unexplained notions.

Since its discovery in the early sixties by Paul Cohen [2], forcing has played a central role in the development of modern set theory. It was soon realized its fundamental role to establish the undecidability in ZFC of all the classical problems of set theory, among which Cantor's continuum problem. Moreover, up to date, forcing (or class forcing) is the unique efficient method to obtain independence results over ZFC. This method has found applications in virtually all fields of pure mathematics: in the last forty years natural problems of group theory, functional analysis, operator algebras, general topology, and many other subjects were shown to be undecidable by means of forcing (see [4, 13] among others). Perhaps driven by these observations Woodin introduced Ω-logic, a non-constructive semantics for ZFC which rules out the independence results obtained by means of forcing.

Definition 1.1. Given a model V of ZFC and a family Γ of partial orders in V, we say that V models that ϕ is Γ-consistent if $V^{\mathbb{B}} \models \phi$ for some $\mathbb{B} \in \Gamma$.

The notions of Γ-validity and of Γ-logical consequence $\models_\Gamma$ are defined accordingly. Woodin's Ω-logic is the Γ-logic obtained by letting Γ be the class of all partial orders[1]. Prima facie Γ-logics appear to be even more intractable than β-logic (the logic given by the class of well founded models of ZFC). However this is a misleading point of view, and, as we shall see below, it is more correct to view these logics as means to radically change our point of view on forcing:

> Γ-logics transform forcing in a tool to prove theorems over certain natural theories T which extend ZFC.

The following corollary of Cohen's forcing theorem (which we dare to call Cohen's Absoluteness Lemma) is an illuminating example:

Lemma 1.2 (Cohen's Absoluteness). *Assume $T \supset \mathsf{ZFC}$ and $\phi(x, r)$ is a Σ_0-formula in the parameter r such that $T \vdash r \subset \omega$. Then the following are equivalent:*

[1] There is a slight twist between Woodin's original definition of Ω-consistency and our definition of Γ-consistency when Γ is the class of all posets. We shall explain in this footnote why we decided to modify Woodin's original definition. On a first reading the reader may skip it over. Woodin states that ϕ is Ω-consistent in V if there is some α and some $\mathbb{B} \in V_\alpha$ such that $V_\alpha^{\mathbb{B}} \models \phi$. The advantage of our definition (with respect to Woodin's) is that it allows for a simpler formulation of the forcing absoluteness results which are the motivation and the purpose of this paper and which assert that over any model V of some theory T which extends ZFC any statement ϕ of a certain form which V models to be Γ-consistent actually holds in V. To appreciate the difference between Woodin's definition of Ω-consistency and the current definition, assume that ϕ is a Π_2-formula and that ϕ is Ω-consistent in V in the sense of Woodin: this means that there exist α and $\mathbb{B}$ such that $V_\alpha^{\mathbb{B}} \models \phi$, nonetheless it is well possible that $V^{\mathbb{B}} \not\models \phi$ and thus that $\mathbb{B}$ does not witness that ϕ is Ω-consistent according to our definition. Now if V models ZFC+*there are class many Woodin cardinals which are a limit of Woodin cardinals* and $\phi^{L(\mathbb{R})}$ is Ω-consistent in V in the sense of Woodin, this can be reflected in the assertion that $\exists \alpha \in V$, $V_\alpha \models \phi^{L(\mathbb{R})}$, but not in the statement that $\phi^{L(\mathbb{R})}$ holds in V. On the other hand if V models ZFC+*there are class many Woodin cardinals which are a limit of Woodin cardinals* and $\phi^{L(\mathbb{R})}$ is Ω-consistent in V according to our definition, we can actually reflect this fact in the assertion that $V \models \phi^{L(\mathbb{R})}$. There is no real discrepancy on the two definitions because for each n we can find a Σ_n formula ϕ_n such that if V is any model of ZF, $V_\alpha \models \phi_n$ if and only if $V_\alpha \prec_{\Sigma_n} V$. Thus, if we want to prove that a certain Σ_n-formula ϕ is Ω-consistent according to our definition, we just have to prove that $\phi_n \wedge \phi$ is Ω-consistent in V according to Woodin's definition. On the other hand the set of Γ-valid statements (according to Woodin's definition) is definable in V in the parameters used to define Γ, while (unless we subsume that there is some δ such that $V_\delta \prec V$ and all the parameters used to define Γ belong to V_δ) we shall encounter the same problems to define in V the class of Γ-valid statements (according to our definition) as we do have troubles to define in V the set of V-truths.

- $T \vdash [H_{\omega_1} \models \exists x\phi(x, r)]$.
- $T \vdash \exists x\phi(x, r)$ is Ω-consistent[2].

Observe that for any model V of ZFC, $H_{\omega_1}^V \prec_{\Sigma_1} V$ and that for any theory $T \supseteq$ ZFC there is a recursive translation of Σ^1_2-properties (provably Σ^1_2 over T) into Σ_1-properties over H_{ω_1} (provably Σ_1 over the same theory T) [6, Lemma 25.25]. Summing up we get that a Σ^1_2-statement is provable in some theory $T \supseteq$ ZFC iff the corresponding Σ_1-statement over H_{ω_1} is provably Ω-consistent over the same theory T. This shows that already in ZFC forcing is an extremely powerful tool to prove theorems. Moreover compare Lemma 1.2 with Shoenfield's absoluteness theorem stating that the truth value of a Σ^1_2-property is the same in all transitive models M of ZFC to which ω_1 belongs [6, Theorem 25.20]. These two results are very similar in nature but the first one is more constructive. For example a proof that a Σ^1_2-property holds in L does not yield automatically that this property is provable in ZFC but just that it holds in all uncountable transitive models of ZFC to which ω_1 belongs; yet this property could fail in some non-transitive model of ZFC or in some transitive model of ZFC whose ordinals have order type at most ω_1.

We briefly sketch why Lemma 1.2 holds since this will outline many of the ideas we are heading for:

Proof. We shall actually prove the following slightly stronger formulation[3] of the non-trivial direction in the equivalence:

> Assume V is a model of T. Then $H_{\omega_1} \models \exists x\phi(x, r)$ if and only if $V \models \exists x\phi(x, r)$ *is* Ω*-consistent*.

To simplify the exposition we prove it with the further assumption that V is a *transitive* model. With the obvious care in details essentially the same argument works for any first order model of T. So assume $\phi(x, \vec{y})$ is a Σ_0-formula and $\exists x\phi(x, \vec{r})$ is Ω-consistent in V with parameters $\vec{r} \in \mathbb{R}^V$. Let $\mathbb{P} \in V$ be a partial order that witnesses it. Pick a model $M \in V$ such that $M \prec (H_{|\mathbb{P}|^+})^V$, M is countable in V, and $\mathbb{P}, \vec{r} \in M$. Let $\pi_M : M \to N$ be its transitive collapse and $\mathbb{Q} = \pi_M(\mathbb{P})$. Notice also that $\pi(\vec{r}) = \vec{r}$. Since π_M is an isomorphism of M with N,

$$N \models (\Vdash_{\mathbb{Q}} \exists x\phi(x, \vec{r})).$$

[2] I.e. $T \vdash$ *There is a partial order* $\mathbb{B}$ *such that* $\Vdash_{\mathbb{B}} \exists x\phi(x, r)$.

[3] In the statement below we do not require that the existence of a partial order witnessing the Ω-consistency of $\exists x\phi(x, r)$ in V is provable in T.

Now let $G \in V$ be N-generic for $\mathbb{Q}$ (G exists since N is countable), then, by Cohen's fundamental theorem of forcing applied in V to N, we have that $N[G] \models \exists x\phi(x, \vec{r})$. So we can pick $a \in N[G]$ such that $N[G] \models \phi(a, \vec{r})$. Since $N, G \in (H_{\aleph_1})^V$, we have that V models that $N[G] \in H^V_{\omega_1}$ and thus V models that a as well belongs to $H^V_{\omega_1}$. Since $\phi(x, \vec{y})$ is a Σ_0-formula, V models that $\phi(a, \vec{r})$ is absolute between the transitive sets $N[G] \subset H_{\omega_1}$ to which $a, \vec{r}$ belong. In particular a witnesses in V that $H^V_{\omega_1} \models \exists x\phi(x, \vec{r})$. $\square$

If we analyze the proof of this Lemma, we immediately realize that a key observation is the fact that for any poset $\mathbb{P}$ there is some countable $M \prec H_{|\mathbb{P}|^+}$ such that $\mathbb{P} \in M$ and there is an M-generic filter for $\mathbb{P}$. The latter statement is an easy outcome of Baire's category theorem and is provable in ZFC. For a given regular cardinal λ and a partial order $\mathbb{P}$, let $S^{\lambda}_{\mathbb{P}}$ be the set consisting of $M \prec H_{\max(|\mathbb{P}|^+, \lambda)}$ such that there is an M-generic filter for $\mathbb{P}$ and $M \cap \lambda \in \lambda > |M|$. Then an easy outcome of Baire's category theorem is that $S^{\aleph_1}_{\mathbb{P}}$ is a club subset of $P_{\omega_1}(H_{|\mathbb{P}|^+})$ for every partial order $\mathbb{P}$. If we analyze the above proof what we actually needed was just the stationarity of $S^{\aleph_1}_{\mathbb{P}}$ to infer the existence of the desired countable model $M \prec H_{|\mathbb{P}|^+}$ such that $r \in M$ and there is an M-generic filter for $\mathbb{P}$. For any regualr cardinal λ, let Γ_λ be the class of posets such that $S^{\lambda}_{\mathbb{P}}$ is stationary. In particular we can generalize Cohen's absoluteness Lemma as follows:

Lemma 1.3 (Generalized Cohen Absoluteness). *Assume V is a model of* ZFC *and λ is regular and uncountable in V. Then $H^V_\lambda \prec_{\Sigma_1} V^P$ if $P \in \Gamma_\lambda$.*

Let $\mathsf{FA}_\nu(\mathbb{P})$ assert that: *P is a partial order such that for every collection of ν-many dense subsets of P there is a filter $G \subset P$ meeting all the dense sets in this collection.* Let $\mathsf{BFA}_\nu(\mathbb{P})$ assert that $H^V_{\nu^+} \prec_{\Sigma_1} V^P$.

Given a class of posets Γ, let $\mathsf{FA}_\nu(\Gamma)$ ($\mathsf{BFA}_\nu(\Gamma)$) hold if $\mathsf{FA}_\nu(P)$ ($\mathsf{BFA}_\nu(P)$) holds for all $P \in \Gamma$. Then Baire's category theorem just says that $\mathsf{FA}_{\aleph_0}(\Omega)$ holds where Ω is the class of all posets. It is not hard to check that if S^{λ}_{P} is stationary, then $\mathsf{FA}_\gamma(P)$ holds for all $\gamma < \lambda$. Woodin [18, Theorem2.53] proved that if $\lambda = \nu^+$ is a successor cardinal $P \in \Gamma_\lambda$ *if and only if* $\mathsf{FA}_\nu(P)$ *holds* (see for more details subsection 2.2 and Lemma 2.9). In particular for all cardinals ν we get that Γ_{ν^+} is the class of partial orders P such that $\mathsf{FA}_\nu(P)$ holds or (equivalently) such that $S^{\nu^+}_P$ is stationary. With this terminology Cohen's absoluteness Lemma states that $\mathsf{FA}_\nu(P)$ implies $\mathsf{BFA}_\nu(P)$ for all infinite cardinals ν.

Observe that many interesting problems of set theory can be formulated as Π_2-properties of H_{ν^+} for some cardinal ν (an example is Suslin's hypothesis, which can be formulated as a Π_2-property of $H_{\aleph_2}$). Lemma 1.3 gives a very powerful

general framework to prove in any given model V of ZFC whether a Π_2-property $\forall x \exists y \phi(x, y, z)$ (where ϕ is Σ_0) holds for some $H^V_{\nu^+}$ with $p \in H^V_{\nu^+}$ replacing z: It suffices to prove that for any $a \in H^V_{\nu^+}$, V models that $\exists y \phi(a, y, p)$ is Γ_{ν^+}-consistent. This shows that if we are in a model V of ZFC where $\Gamma^V_{\nu^+}$ contains interesting and manageable families of partial orders $\Gamma^V_{\nu^+}$-logic is a powerful tool to study the Π_2-theory of $H^V_{\nu^+}$. In particular this is always the case for $\nu = \aleph_0$ in any model of ZFC, since $\Gamma_{\aleph_1}$ is the class of all posets. Moreover this is certainly one of the reasons of the success the forcing axiom Martin's Maximum MM and its bounded version BMM have had in settling many relevant problems of set theory which can be formulated as Π_2-properties of the structure $H_{\aleph_2}$ and that boosted the study of bounded versions of forcing axioms[4].

For any set theorist willing to accept large cardinal axioms, Woodin has been able to show that Ω-logic gives a natural non-constructive semantics for the full first order theory of $L(\mathbb{R})$ and not just for the Σ_1-fragment of $H_{\aleph_1} \subset L(\mathbb{R})$ which is given by Cohen's absoluteness Lemma. Woodin [10, Theorem 2.5.10] has proved that assuming large cardinals Ω-*truth is* Ω-*invariant* i.e.:

> Let V be any model of ZFC+*there are class many Woodin cardinals*.
> Then for any statement ϕ with parameters in $\mathbb{R}^V$,
>
> $$V \models (\phi \text{ is } \Omega\text{-consistent})$$
>
> if and only if there is $\mathbb{B} \in V$ such that
>
> $$V^{\mathbb{B}} \models (\phi \text{ is } \Omega\text{-consistent}).$$

Thus Ω-logic, the logic of forcing, has a notion of truth which forcing itself cannot change. Woodin [10, Theorem 3.1.7] also proved that the theory ZFC+*large cardinals* decides in Ω-logic the theory of $L(\mathbb{R})$, i.e.:

> For any model V of ZFC+*there are class many Woodin cardinals which are a limit of Woodin cardinals* and any first order formula ϕ, $L(P_{\omega_1}\mathrm{Ord})^V \models \phi$ if and only if
>
> $$V \models [L(P_{\omega_1}\mathrm{Ord}) \models \phi] \text{ is } \Omega\text{-consistent.}$$

[4]Bagaria [1] and Stavi, Väänänen [14] are the first who realize that bounded forcing axioms are powerful tools to describe the Π_2-theory of $H_{\aleph_2}$ exactly for the reasons we are pointing out.

He pushed further these result and showed that if T extends ZFC+ *There are class many measurable Woodin cardinals*, then T decides in Ω-logic any mathematical problem expressible as a (provably in T) Δ^2_1-statement. These are optimal and sharp results: it is well known that the Continuum hypothesis CH (which is provably not a Δ^2_1-statement) and the first order theory of $L(P(\omega_1))$ cannot be decided by ZFC+ *large cardinal axioms* in Ω-logic. Martin and Steel's result that projective determinacy holds in ZFC* complements the fully satisfactory description Ω-logic and large cardinals give of the first order theory of the structure $L(\mathbb{R})$ in models of ZFC*. Moreover we can make these results meaningful also for a non-platonist, for example we can reformulate the statement that ZFC* decides in Ω-logic the theory of $L(\mathbb{R})$ as follows:

> Assume T extends ZFC+*there are class many Woodin cardinals which are a limit of Woodin cardinals*. Let $\phi(r)$ be a formula in the parameter r such that $T \vdash r \subseteq \omega$. Then the following are equivalent:
>
> - $T \vdash [L(P_{\omega_1}\mathrm{Ord}) \models \phi(r)]$.
> - $T \vdash \phi(r)^{L(P_{\omega_1}\mathrm{Ord})}$ *is Ω-consistent.*

The next natural stage is to determine to what extent Woodin's results on Ω-logic and the theory of $H_{\aleph_1}$ and $L(\mathbb{R})$ can be reproduced for $H_{\aleph_2}$ and $L(P(\omega_1))$. There is also for these theories a fundamental result of Woodin: he introduced an axiom $(*)$ which is a strengthened version of BMM with the property that the theory of $H_{\aleph_2}$ with parameters is invariant with respect to *all* forcings which preserve this axiom[5]. The $(*)$-axiom is usually formulated [9, Definition 7.9] as the assertion that $L(\mathbb{R})$ is a model of the axiom of determinacy and $L(P(\omega_1))$ is a generic extension of $L(\mathbb{R})$ by an homogeneous forcing $\mathbb{P}_{\max} \in L(\mathbb{R})$.

There are two distinctive features of $(*)$:

1. It asserts a smallness principle for $L(P(\omega_1))$: on the one hand the homogeneity of $\mathbb{P}_{\max}$ entails that the first order theory of $L(P(\omega_1))$ is essentially determined by the theory of the underlying $L(\mathbb{R})$. On the other hand $(*)$ implies that $L(P(\omega_1)) = L(\mathbb{R})[A]$ for any $A \in P(\omega_1) \setminus L(\mathbb{R})$.

2. $(*)$ entails that $H^V_{\omega_2} \prec H^{V^P}_{\omega_2}$ for any notion of forcing $P \in V$ which preserves $(*)$ even if $\mathsf{FA}_{\aleph_2}(P)$ may be false for such a P.

[5] We refer the reader to [9] for a thorough development of the properties of models of the $(*)$-axiom.

In this paper we propose a different approach to the analysis of the theory of $H_{\aleph_2}$ then the one given by $(*)$. We do not seek for an axiom system $T \supseteq \mathsf{ZFC}$ which makes the theory of $H_{\aleph_2}$ invariant with respect to *all* forcing notions which preserve a suitable fragment of T. Our aim is to show that the strongest forcing axioms in combination with large cardinals give an axiom system T which extends ZFC and makes the theory of $H_{\aleph_2}$ invariant with respect to all forcing notions P which preserve a suitable fragment of T *and for which we can predicate* $\mathsf{FA}_{\aleph_1}(P)$ (i.e. forcings P which are in the class $\Gamma_{\aleph_2}$).

This leads us to analyze the properties of the class $\Gamma_{\aleph_2}$ in models of ZFC^*. This is a delicate matter, first of all Shelah proved that $\mathsf{FA}_{\aleph_1}(P)$ fails for any P which does not preserve stationary subsets of ω_1. Nonetheless it cannot be decided in ZFC whether this is a necessary condition for a poset P in order to have the failure of $\mathsf{FA}_{\aleph_1}(P)$. For example let P be Namba forcing: it is provable in ZFC that P preserve stationary subsets of ω_1, however in L $\mathsf{FA}_{\aleph_1}(P)$ fails while in a model of Martin's maximum MM $\mathsf{FA}_{\aleph_1}(P)$ holds. This shows that we cannot hope to prove general theorems about $H_{\aleph_2}$ in ZFC^* alone using forcing, but just theorems about the properties of $H_{\aleph_2}$ for particular theories T which extend ZFC^* and for which we have a nice description of the class $\Gamma_{\aleph_2}$.

In this respect it is well known that the study of the properties of $H_{\aleph_2}$ in models of Martin's maximum MM, of the proper forcing axiom PFA, or of their bounded versions BMM and BPFA has been particularly successful. Moreover it is well known that the strongest such theories (MM and PFA) are able to settle many relevant questions about the whole universe V and to show that many properties of the universe reflect to $H_{\aleph_2}$[6]. The reason is at least two-fold:

- First of all there is a manageable description of the class $\Gamma_{\aleph_2}$ in models of MM (PFA,MA): this is the class of stationary set preserving posets for MM (respectively contains the class of proper forcings for PFA, and the class of CCC partial orders for MA).

- MM realizes the slogan that $\mathsf{FA}_{\aleph_1}(P)$ holds for any partial order P for which we cannot prove that $\mathsf{FA}_{\aleph_1}(P)$ fails, thus MM substantiates a natural maximality principle for the class $\Gamma_{\aleph_2}$.

[6] The literature is vast, we mention just a sample of the most recent results with no hope of being exhaustive: [11, 15, 18] present different examples of well-ordering of the reals definable in $H_{\aleph_2}$ (with parameters in $H_{\aleph_2}$) in models of BMM (BPFA), [3, 16, 17] present several different reflection properties between the universe and $H_{\aleph_2}$ in models of MM^{++} (PFA,MM), [4, 12] present applications of PFA to the solution of problems coming from operator algebra and general topology and which can be formulated as (second order) properties of the structure $H_{\aleph_2}$.

We believe that the arguments we presented so far already show that for any model V of ZFC and any successor cardinal $\lambda \in V$ it is of central interest to analyze what is the class Γ_λ in V, since this gives a powerful tool to ivestigate the Π_2-theory of H_λ^V. Moreover in this respect $\mathsf{ZFC} + \mathsf{MM}$ is particularly appealing since it asserts the maximality of the class $\Gamma_{\aleph_2}$. The main result of this paper is to show that a natural strengthening of MM (denoted by MM^{++}) which holds in the standard models of MM, in combination with Woodin cardinals, makes $\Gamma_{\aleph_2}$-logic the correct semantics to describe completely the Π_2-theory of $H_{\aleph_2}$ in models of MM^{++}. In particular we shall prove the following theorem:

Theorem 1.4. *Assume* MM^{++} *holds and there are class many Woodin cardinals. Then*

$$H_{\aleph_2}^V \prec_{\Sigma_2} H_{\aleph_2}^{V^P}$$

for all stationary set preserving posets P which preserve BMM

Notice that we can reformulate the theorem in the same fashion of Woodin's and Cohen's results as follows:

Theorem 1.5. *Assume T extends* $\mathsf{ZFC} + \mathsf{MM}^{++}+$ There are class many Woodin cardinals. *Then for every Π_2-formula $\phi(x)$ in the free variable x and every parameter p such that $T \vdash p \in H_{\omega_2}$ the following are equivalent:*

- $T \vdash [H_{\aleph_2} \models \phi(p)]$
- $T \vdash$ There is a stationary set preserving partial order P such that $\Vdash_P \phi^{H_{\aleph_2}}(p)$ and $\Vdash_P \mathsf{BMM}$.

The rest of this paper is organized as follows: Section 2 presents background material on forcing (Subsection 2.1), forcing axioms (Subsection 2.2), the stationary tower forcing (Subsection 2.3), the relation between the stationary tower forcing and forcing axioms (Subsection 2.4), and a new characterization of the forcing axiom MM^{++} in terms of complete embeddings of stationary set preserving posets into stationary tower forcings (Subsection 2.5). Section 3 presents the proof of the main result, while Section 4 gives a proof of the invariance of the theory of $H_{\aleph_1}$ with respect to set forcing in the presence of class many Woodin cardinals. We end the paper with some comments and open questions (Section 5).

While the paper is meant to be as much self-contained as possible, we presume that familiarity with forcing axioms (in particular with Martin's maximum) and with the stationary tower forcing are of valuable help for the reader. A good reference for background material on Martin's maximum is [6, Chapter 37]. For the stationary tower forcing a reference text is [10].

1.1 Notation and prerequisites

We adopt standard notation which is customary in the subject, our reference text is [6].

For models (M, E) of ZFC, we say that $(M, E) \prec_{\Sigma_n} (M', E')$ if $M \subset M'$, $E = E' \cap M^2$ and for any Σ_n-formula $\phi(p)$ with $p \in M$, $(M, E) \models \phi(p)$ if and only if $(M', E') \models \phi(p)$. We usually write $M \prec_{\Sigma_n} M$ instead of $(M, E) \prec_{\Sigma_n} (M', E')$ when E, E' is clear from the context. We let $(M, E) \prec (M', E')$ if $(M, E) \prec_{\Sigma_n} (M', E')$ for all n.

We let Ord denote the class of ordinals. For any cardinal κ $P_\kappa X$ denote the subsets of X of size less than κ. Given $f : X \to Y$ and $A \subset X, B \subset Y$, $f[A]$ is the pointwise image of A under f and $f^{-1}[B]$ is the preimage of B under f. A set S is stationary if for all $f : P_\omega(\cup S) \to \cup S$ there is $X \in S$ such that $f[X] \subseteq X$ (such an X is called a closure point for f). A set C is a club subset of S if it meets all stationary subsets of S or, equivalently, if it contains all the closure points in S of some $f : P_\omega(\cup S) \to \cup S$. Notice that $P_\kappa X$ is always stationary if κ is a cardinal and X, κ are both uncountable.

If V is a transitive model of ZFC and $(P, \leq_P) \in V$ is a partial order with a top element 1_P, V^P denotes the class of P-names, and $\dot{a}$ or τ denote an arbitrary element of V^P, if $\check{a} \in V^P$ is the canonical name for a set a in V we drop the superscript and confuse $\check{a}$ with a. We also feel free to confuse the approach to forcing via boolean valued models as done by Scott and others or via the forcing relation. Thus we shall write for example $V^P \models \phi$ as an abbreviation for

$$V \models [1_P \Vdash \phi].$$

If $M \in V$ is such that $(M, \in)$ is a model of a sufficient fragment of ZFC and $(P, \leq_P)$ in M is a partial order, an M-generic filter for P is a filter $G \subset P$ such that $G \cap A \cap M$ is non-empty for all maximal antichains $A \in M$ (notice that if M is non-transitive, $A \nsubseteq M$ is well possible). If N is a *transitive* model of a large enough fragment of ZFC, $P \in N$ and G is an N-generic filter for P, let $\sigma_G : N \cap V^P \to N[G]$ denote the evaluation map induced by G of the P-names in N.

We say that $(M, E) \prec_{\Sigma_n} (\dot{N}, \dot{E})$ for some $\dot{N} \in V^P$ if

$$V^P \models \dot{E} \cap M^2 = E$$

and for any Σ_n-formula $\phi(p)$ with $p \in M$, $(M, E) \models \phi(p)$ if and only if

$$V^P \models [(\dot{N}, \dot{E}) \models \phi(p)].$$

We will write $M \prec_{\Sigma_n} \dot{N}$ if $(M, E) \prec_{\Sigma_n} (\dot{N}, \dot{E})$ and $E, \dot{E}$ are clear from the context.

We shall also frequently refer to Woodin cardinals, however for our purposes we won't need to recall the definition of a Woodin cardinal but just its effects on the properties of the stationary tower forcing. This is done in subsection 2.3.

2 Preliminaries

We shall briefly outline some general results on the theory of forcing which we shall need for our exposition. The reader may skip Subsections 2.1, 2.2, 2.3 and eventually refer back to them.

2.1 Preliminaries I: complete embeddings and projections

For a poset Q and $q \in Q$, let $Q \restriction q$ denote the poset Q restricted to conditions $r \in Q$ which are below q and $\mathbb{B}(Q)$ denote its boolean completion, i.e. the complete boolean algebra of regular open subsets of Q, so that Q is naturally identified with a dense subset of $\mathbb{B}(Q)$. We say that:

- P completely embeds into Q if there is an homomorphism $i : P \to \mathbb{B}(Q)$ which preseves the order relation and maps maximal antichains of P into maximal antichains of $\mathbb{B}(Q)$. With abuse of notation we shall call a complete embedding of P into Q any such homomorphism $i : P \to \mathbb{B}(Q)$.

- $i : P \to \mathbb{B}(Q)$ is locally complete if for some $q \in Q$, $i : P \to \mathbb{B}(Q \restriction q)$ is a complete embedding (with a slight abuse of the current terminology, we shall also call any locally complete embedding a *regular* embedding).

- P projects to Q if there is an order preserving map $\pi : P \to Q$ whose image is dense in Q.

Lemma 2.1. *The following are equivalent:*

1. *P completely embeds into Q,*

2. *for any V-generic filter G for Q there is in $V[G]$ a V-generic filter H for P,*

3. *For some $p \in P$ there is a homomorphism $i : \mathbb{B}(P \restriction p) \to \mathbb{B}(Q)$ of complete boolean algebras.*

Proof. We proceed as follows:

1 implies 2

Observe that if $i : P \to \mathbb{B}(Q)$ is a complete embedding and G is a V-generic filter for $\mathbb{B}(Q)$, then $H = i^{-1}[G]$ is a V-generic filter for P.

2 implies 1

Let $\dot{H} \in V^{\mathbb{B}(Q)}$ be a name such that

$$\Vdash_{\mathbb{B}(Q)} \dot{H} \text{ is a } V\text{-generic filter for } P.$$

The map $p \mapsto \|\check{p} \in \dot{H}\|_{\mathbb{B}(Q)}$ is the desired complete embedding of P into Q.

1 implies 3

Let $i : P \to \mathbb{B}(Q)$ be a complete embedding and $\dot{H} \in V^{\mathbb{B}(Q)}$ be a name for the V-generic filter for $\mathbb{B}(Q)$. Then there is some $p \in P$ such that

$$\|i(q) \in \dot{H}\|_{\mathbb{B}(Q)} > 0_{\mathbb{B}(Q)}$$

for all $q \leq p$. Then for such a p the map i can naturally be extended to a complete homomorphism $i : \mathbb{B}(P \restriction p) \to \mathbb{B}(Q)$.

3 implies 1

Immediate. $\square$

Remark 2.2. Observe that if $i : P \to \mathbb{B}(Q)$ is a complete embedding then for all $q \in Q$ such that $i(p) \wedge q > 0_{\mathbb{B}}$, the map $i_q : P \to \mathbb{B}(Q \restriction q)$ which maps p to $q \wedge i(p)$ is also a complete embedding. Moreover if $q \Vdash_Q \check{p} \in \dot{H}$ where $\dot{H} = i^{-1}[\dot{G}] \in V^Q$ and $\dot{G}$ is the canonical $\mathbb{B}(Q)$-name for a V-generic filter for $\mathbb{B}(Q)$, we have that $i_q(r) = 0_{\mathbb{B}(Q)}$ for all $r \in P$ incompatible with p.

Thus in general a complete embedding $i : P \to \mathbb{B}(Q)$ may map a large portion of P to $0_{\mathbb{B}(Q)}$.

Lemma 2.3. *The following are equivalent:*

1. *There is a projection* $\pi : P \to \mathbb{B}(Q) \setminus \{0_{\mathbb{B}(Q)}\}$.
2. *There is* $\dot{H}$ *in* V^P *such that* $\Vdash_P \dot{H}$ is a V generic filter for Q.

Proof. **1 implies 2**

Let $\dot{H} \in V^P$ be a P-name such that $\Vdash_P \dot{H} = \pi[\dot{G}]$. Then since the image of π is a dense subset of $\mathbb{B}(Q)$ it is easy to check that $\Vdash_P$ $\dot{H}$ *generates a* V*-generic filter for* $\mathbb{B}(Q)$.

2 implies 1 Assume 2 holds for the P-name $\dot{H}$ and let $\pi : P \to \mathbb{B}(Q)$ be defined by $\pi(p) = \bigwedge\{q \in Q : p \Vdash_P q \in \dot{H}\}$. We claim that π is a projection. First of all we claim that $\pi(p) > 0_{\mathbb{B}(Q)}$ for all $p \in P$. This uses the following observation:

Fact 2.4. Assume G is V-generic for P and $H = \sigma_G(\dot{H}) \in V[G]$ is V-generic for Q. If $A \in V$ is such that $A \subset H$, then $\bigwedge A > 0_{\mathbb{B}(Q)}$.

Proof. Assume not, then there is some $r \in G$ such that $r \Vdash_P \bigwedge A = 0_{\mathbb{B}(Q)}$. Now let $A = \{a_i : i \in I\}$. Since $A \subset G$ all the a_i are compatible. Let $B \subset A$ $B \in V$ be a non-empty subset of A of least size such that $\bigwedge B = 0_{\mathbb{B}(Q)}$ but for no $E \subset B$ such that $|E| < |B|$, $\bigwedge E = 0_{\mathbb{B}(Q)}$. Then we can arrange that $B = \{a_\alpha : \alpha < \gamma\}$ for some cardinal γ and that $b_\beta = \bigwedge\{a_\alpha : \alpha < \beta\} > 0_{\mathbb{B}(Q)}$ for all $\beta < \gamma$. By refining the sequence $\{b_\beta : \beta < \gamma\}$ (if necessary) we can further suppose that $b_\alpha < b_\beta$ for all $\alpha < \beta$.

Now let $c_\beta = b_0 \wedge \neg b_\beta$. We claim that

$$C = \{c : \exists\beta < \gamma \text{ such that } c \leq c_\beta\} \qquad \square$$

(which belongs to V) is a dense subset of $\mathbb{B}(Q) \restriction b_0$. To see this, let $\dot{H}_0$ be a P-name which is forced by P to be the V-generic filter for $\mathbb{B}(Q)$ generated by $\dot{H}$ which is a P-name for a V-generic filter for Q. Observe that since $r \Vdash_P$ $\dot{H}_0$ *is a* V*-generic filter for* $\mathbb{B}(Q)$ *containing* $\dot{H}$, we get that $r \Vdash_P b_\beta \in \dot{H}_0$ for all $\beta < \gamma$. Now given some $c \leq b_0$, we have that $\bigwedge\{c \wedge b_\beta : \beta < \gamma\} = 0_{\mathbb{B}(Q)}$. There are two cases:

- There is some $\beta < \gamma$ such that $c \wedge b_\delta = c \wedge b_\beta$ for all $\delta \in [\beta, \gamma)$.

  In this case we have that $c \wedge b_\beta = 0_{\mathbb{B}(Q)}$ and c is already an element of C.

- For all $\beta < \gamma$ there is $\alpha_\beta < \gamma$ such that $c \wedge b_{\alpha_\beta} < c \wedge b_\beta$.

  In this case we get that $c \wedge b_{\alpha_0} < c$. Thus $d = c \wedge \neg b_{\alpha_0} > 0_{\mathbb{B}(Q)}$ and $d \leq c$ is an element of C.

Thus C is dense. Since $r \Vdash_P C \cap \dot{H}_0 \neq \emptyset$ we can find $r' \leq r$ and $d \in C$ such that $r' \Vdash_P d \in C \cap \dot{H}_0$. Now there is some β such that $d \wedge b_\beta = 0_{\mathbb{B}(Q)}$. Then $r' \Vdash_P d \wedge b_\beta = 0_{\mathbb{B}(Q)} \in \dot{H}_0$, which is the desired contradiction which proves the fact.

Now for all $p \in P$, $A_p = \{q \in Q : p \Vdash q \in \dot{H}\}$ is in V and is forced by p to be a subset of $\dot{H}$. In particular we get that $\bigwedge A_p = \pi(p) > 0_{\mathbb{B}(Q)}$. iI is now easy to check that $\pi[P]$ is a dense subset of $\mathbb{B}(Q) \setminus \{0_{\mathbb{B}(Q)}\}$. $\square$

Given a complete embedding $i : \mathbb{Q} \to \mathbb{B}$ of complete boolean algebras, let $\pi : \mathbb{B} \to \mathbb{Q}$ map a to $\inf\{q \in \mathbb{Q} : i(q) \geq a\}$, then π is a projection and $\pi \circ i(b) = b$ for all $b \in \mathbb{B}$ while $i \circ \pi(q) \geq q$ for all $q \in \mathbb{Q}$.

The quotient forcing $\mathbb{B}/i[\mathbb{Q}]$ is the object belonging to $V^{\mathbb{Q}}$ such that

- $\Vdash_{\mathbb{Q}} \mathbb{B}/i[\mathbb{Q}]$ *is a partial order with the order relation* $\leq_i$.
- $\mathbb{B}/i[\mathbb{Q}] \in V^{\mathbb{Q}}$ is the set of $\mathbb{Q}$-names $\dot{r}$ of least rank among those that satisfy the following requirements:
  - $[\![\dot{r} \in (\check{\mathbb{B}} \setminus \{0_{\mathbb{B}}\})]\!]_{\mathbb{Q}} = 1_{\mathbb{Q}}$.
  - For all $\dot{r} \in \mathbb{B}/i[\mathbb{Q}]$ if there are $r \in \mathbb{B}$ and $q \in \mathbb{Q}$ such that $q \Vdash_{\mathbb{Q}} \dot{r} = r$, then $\pi(r) \geq q$.
- For $\dot{r}, \dot{s} \in \mathbb{B}/i[\mathbb{Q}]$ $q \Vdash_{\mathbb{Q}} \dot{r} \leq_i \dot{s}$ if and only if the following holds:

  For all $q' \leq q$, if there are $r, s \in \mathbb{B}$ such that $q' \Vdash_{\mathbb{Q}} \dot{r} = r \wedge \dot{s} = s$, then $r \wedge i(q') \leq_{\mathbb{B}} s \wedge i(q')$.

Lemma 2.5. *If $i : \mathbb{Q} \to \mathbb{B}$ is a complete embedding of complete boolean algebras, then $\mathbb{Q} * (\mathbb{B}/i[\mathbb{Q}])$ is forcing equivalent to $\mathbb{B}$.*

Proof. Let $\pi : \mathbb{B} \to \mathbb{Q}$ be the projection map associated to i. The map

$$i^* : (\mathbb{B} \setminus \{0_{\mathbb{B}}\}) \to \mathbb{B}(\mathbb{Q} * (\mathbb{B}/i[\mathbb{Q}]))$$

which maps $r \mapsto (\pi(r), \check{r})$ is a complete embedding such that $i^*[\mathbb{B} \setminus \{0_{\mathbb{B}}\}]$ is dense in $\mathbb{Q} * (\mathbb{B}/i(\mathbb{Q}))$.

The conclusion follows. $\square$

Remark 2.6. There might be a variety of locally complete embeddings of a poset P into a poset Q. These embeddings greatly affect the properties the generic extensions by Q attributes to elements of the generic extensions by P. For example the following can occur:

> There is a P-name $\dot{S}$ which is forced by P to be a stationary subset of ω_1 and there are $i_0 : P \to \mathbb{B}(Q)$, $i_1 : P \to \mathbb{B}(Q)$ distinct locally complete embeddings of P into Q such that if G is V-generic for $\mathbb{B}(Q)$ and $H_j = i_j^{-1}[G]$, then $\sigma_{H_0}(\dot{S})$ is stationary in $V[G]$, $\sigma_{H_1}(\dot{S})$ is stationary in $V[H_1]$ but non-stationary in $V[G]$.

If $i : P \to \mathbb{B}(Q)$ is a locally complete embedding and $p \in P$, $q \in Q$ are such that i can be extended to a complete homomorphism of $\mathbb{B}(P \restriction p)$ into $\mathbb{B}(Q \restriction q)$ we shall also denote $\mathbb{B}(Q \restriction q)/i[\mathbb{B}(P \restriction p)]$ by $Q/i[P]$, if i is clear from the context we shall even denote such quotient forcing as Q/P.

2.2 Preliminaries II: forcing axioms

Definition 2.7. Given a cardinal λ and a partial order P, $\mathsf{FA}_\lambda(P)$ holds if:

> For all $p \in P$, $P \restriction p$ is a partial order such that for every collection of λ-many dense subsets of $P \restriction p$ there is a filter $G \subset P \restriction p$ meeting all the dense set in this collection.

$\mathsf{FA}_{<\lambda}(P)$ holds if $\mathsf{FA}_\nu(P)$ holds for all $\nu < \lambda$.

$\mathsf{BFA}_\lambda(P)$ holds if $H_\lambda \prec_{\Sigma_1} V^P$.

If Γ is a family of partial orders, $\mathsf{FA}_\lambda(\Gamma)$ ($\mathsf{FA}_{<\lambda}(\Gamma)$, $\mathsf{BFA}(\Gamma)$) asserts that $\mathsf{FA}_\lambda(P)$ ($\mathsf{FA}_{<\lambda}(P)$, $\mathsf{BFA}(P)$) holds for all $P \in \Gamma$.

For any partial order P

$$S^\lambda_P = \{M \prec H_{|P|^+} : M \cap \lambda \in \lambda > |M| \text{ and there is an } M\text{-generic filter for } P\}$$

For any regular uncountable cardinal λ, we let Γ_λ be the family of P such that S^λ_P is stationary.

In the introduction we already showed:

Lemma 2.8. *Assume λ is a regular cardinal. Then $P \in \Gamma_\lambda$ implies* $\mathsf{BFA}_\lambda(P)$.

MM asserts that $\mathsf{FA}_{\aleph_1}(\mathsf{SSP})$ holds, where SSP is the family of posets which preserve stationary subsets of ω_1. BMM asserts that $\mathsf{BFA}_{\aleph_1}(\mathsf{SSP})$ holds. It is not hard to see that if S_P^λ is stationary, then $\mathsf{FA}_{<\lambda}(P)$ holds. It is not clear whether the converse holds if λ is inaccessible. However the converse holds if λ is a successor cardinal and Woodin's [18, Theorem 2.53] gives a special case of the following Lemma for $\lambda = \omega_2$.

Lemma 2.9. *Let $\lambda = \nu^+$ be a successor cardinal. Then the following are equivalent:*

1. $\mathsf{FA}_\nu(P)$ *holds.*
2. S_P^λ *is stationary.*

Proof. Only one direction is non trivial. We assume that $\mathsf{FA}_\nu(P)$ holds in V and we prove that V models that S_P^λ is stationary. We leave to the reader to prove the other implication.

First of all we leave the reader to check that if $\mathsf{FA}_\nu(P)$ holds, then all cardinals less or equal to ν are preserved by P. Let $P \in H_\theta$ with θ regular larger than λ.

Pick $M_0 \prec H_\theta$ such that $P \in M_0$ and $M_0 \cap \lambda \in \lambda > |M_0| = \nu$. Now since $|M_0| = \nu$, there is a filter H which meets all the dense sets in M_0. The proof is completed once we prove the following:

Claim 2.9.1.

$$M_1 = \{a \in H_\theta : \exists \tau \in M_0 \cap V^P \exists q \in H \text{ such that } q \Vdash_P a = \tau\} \prec H_\theta,$$

H is an M_1-generic filter for P, $|M_1| = \nu$ and $M_1 \cap \lambda \in \lambda$.

Proof. We prove each item as follows:

- **$M_1 \prec H_\theta$:**

  Given a first order formula $\phi(x_0, \ldots, x_n)$, and $a_1, \ldots a_n \in M_1$ such that H_θ models $\exists x \phi(x, a_1 \ldots, a_n)$ we want to find $a_0 \in M_1$ such that H_θ models $\phi(a_0, a_1 \ldots, a_n)$. Let $\tau_1, \ldots, \tau_n \in M_0 \cap V^P$ be such that for some $q_i \in H$, $q_i \Vdash \tau_i = a_i$ and
  $$\Vdash_P \exists x \in H_\theta^V \phi(x, \tau_1, \ldots \tau_n)^{H_\theta^V}.$$
  Since $P \in H_\theta$ we can find $\tau \in H_\theta$ such that
  $$\Vdash_P \phi(\tau, \tau_1, \ldots \tau_n)^{H_\theta^V} \wedge \tau \in V.$$

In particular we get that

$$H_\theta \models [\Vdash_P \tau \in V \wedge \phi(\tau, \tau_1, \ldots, \tau_n)^{H_\theta^V}].$$

Since $M_0 \prec H_\theta$, we can actually find such a $\tau \in M_0 \cap V^P$. Then the set of $q \in P$ which force the value of τ to be some element of H_θ is open dense and belongs to M_0. Thus there is $q \in H$ which belongs to this open dense set and refines all the q_i, and $a \in H_\theta$ such that $q \Vdash a = \tau$. Then $a \in M_1$ and H_θ models that $\phi(a, a_1 \ldots, a_n)$, as was to be shown.

- **H is an M_1-generic filter for P:**

  Pick $D \in M_1$ dense subset of P and $\dot{D} \in M_0$ such that $\Vdash_P$ τ *is a dense subset of P which belongs to V* and such that for some $q \in H$, $q \Vdash_P \dot{D} = D$. Then we get that $\Vdash_P \tau \cap \dot{G} \neq \emptyset$, thus there is some $\tau' \in M_0$ such that $\Vdash_P \tau' \in \dot{D} \cap \dot{G}$, since $M_0 \prec H_\theta$. Now we can find $r \leq q$, $r \in H$ and $p \in P$ such that $r \Vdash_P \tau' = p$. Since

  $$r \Vdash_P p = \tau' \in \dot{G} \cap \dot{D} = \dot{G} \cap D,$$

  we get that $p \geq r$ is also in H and thus that $H \cap D \neq \emptyset$.

- **$M_1 \cap \lambda \in \lambda > |M_1| = \nu$:**

  First of all M_1 has size $|M_0| = \nu$ since it is the surjective image of $M_0 \cap V^P$ and contains M_0. Thus $\sup(M_1 \cap \lambda) < \lambda$. Now pick $\beta \in M_1 \cap \lambda$. Find $\tau \in M_0$ such that $\Vdash_P \tau \in \lambda$ and for some $q \in H$, $q \Vdash_P \tau = \beta$. Let $\phi_\tau \in V^P \cap M_0$ be a P-name such that $\Vdash_P \phi_\tau : \nu \to \tau$ *is a bijection which belongs to V*. Find $r \leq q$ $r \in H$ such that $r \Vdash \phi_\tau = \phi$ for some $\phi \in V$ bijection of ν with β. Since $\nu \subset M_0 \subset M_1$ we get that $\phi[\nu] = \beta \subset M_1$. □

The Claim and thus the Lemma are proved (Notice that the unique part of the proof in which we used that λ is a successor cardinal is to get that $M_1 \cap \lambda \in \lambda$). □

2.3 Preliminaries III: stationary sets and the stationary tower forcing

S is stationary if for all $f : P_\omega(\cup S) \to (\cup S)$ there is an $X \in S$ such that $f[P_\omega(X)] \subset X$.

For a stationary set S and a set X, if $\cup S \subseteq X$ we let $S^X = \{M \in P(X) : M \cap \cup S \in S\}$, if $\cup S \supseteq X$ we let $S \restriction X = \{M \cap X : M \in S\}$.

If S and T are stationary sets we say that S and T are compatible if

$$S^{(\bigcup S)\cup(\bigcup T)} \cap T^{(\bigcup S)\cup(\bigcup T)}$$

is stationary.

We let $S \wedge T$ denote the set of $X \in P(\cup S \cup \cup T)$ such that $X \cap \cup S \in S$ and $X \cap \cup T \in T$ and for all η $\bigwedge\{S_\alpha : \alpha < \eta\}$ is the set of $M \in P(\bigcup_{\alpha<\eta} S_\alpha)$ such that $M \cap \cup S_\alpha \in S_\alpha$ for all $\alpha \in M \cap \eta$.

For a set M we let $\pi_M : M \to V$ denote the transitive collapse of the structure $(M, \in)$ onto a transitive set $\pi_M[M]$ and we let $j_M = \pi_M^{-1}$.

For any regular cardinal λ

$$R_\lambda = \{X : X \cap \lambda \in \lambda \text{ and } |X| < \lambda\}.$$

and $\mathbb{R}^\lambda_\delta$ is the stationary tower whose elements are stationary sets $S \in V_\delta$ such that $S \subset R_\lambda$ with order given by $S \leq T$ if, letting $X = \cup(T) \cup \cup(S)$, S^X is contained in T^X modulo a club.

$\mathbb{R}_\delta$ will denote $\mathbb{R}^{\aleph_2}_\delta$.

We recall that if G is V-generic for $\mathbb{R}^\lambda_\delta$, then G induces in a natural way a direct limit ultrapower embedding $j_G : V \to M^G$ where $[f]_G \in M^G$ if $f : P(X_f) \to V$ in V and $[f]_G\ R_G\ [h]_G$ iff for some $\alpha < \delta$ such that $X_f, X_h \in V_\alpha$ we have that

$$\{M \prec V_\alpha : f(M \cap X_f)\ R\ h(M \cap X_h)\} \in G.$$

If M^G is well founded it is customary to identify M^G with its transitive collapse.

We recall the following results about the stationary tower (see [10, Chapter 2]):

Theorem 2.10 (Woodin). *Assume δ is a Woodin cardinal, $\lambda < \delta$ is regular and G is V-generic for $\mathbb{R}^\lambda_\delta$. Then*

1. *M^G is a definable class in $V[G]$ and*

$$V[G] \models (M^G)^{<\delta} \subseteq M^G.$$

2. *$V_\delta, G \subseteq M^G$ and $j_G(\lambda) = \delta$.*

3. $M^G \models \phi([f_1]_G, \ldots, [f_n]_G)$ *if and only for some* $\alpha < \delta$ *such that* $f_i : P(X_i) \to V$ *are such that* $X_i \in V_\alpha$ *for all* $i \leq n$*:*

$$\{M \prec V_\alpha : V \models \phi(f_1(M \cap X_1), \ldots, f_n(M \cap X_n))\} \in G.$$

Moreover by 1 M^G *is well founded and thus can be identified with its transitive collapse. With this identifications we have that for every* $\alpha < \delta$ *and every set* $X \in V_\alpha$, $X = [\langle \pi_M(X) : M \prec V_\alpha, X \in M \rangle]_G$. *In particular with this identification we get that*

$$(H_{j_G(\lambda)})^{M[G]} = V_\delta[G] = (H_\delta)^{V[G]}.$$

and that $j_G \restriction H^V_\lambda$ *is the identity and witnesses that* $H^V_\lambda \prec H^{V[G]}_{j_G(\lambda)}$.

2.4 Preliminaries IV: Woodin cardinals are forcing axioms

The following is an outcome of Woodin's work on the stationary tower [18, Theorem 2.53].

Lemma 2.11 (Woodin). *Assume there are class many Woodin cardinals. and* λ *is a regular cardinal. Then the following are equivalent:*

1. $S^{\lambda}_{\mathbb{P}}$ *is stationary, where*

$$S^{\lambda}_{\mathbb{P}} = \{M \prec H_{|\mathbb{P}|^+} : M \in R_\lambda \text{ and there is an } M\text{-generic filter for } \mathbb{P}\}$$

2. $\mathbb{P}$ *completely embeds into* $\mathbb{R}^\lambda_\delta \restriction T$ *for some Woodin cardinal* δ *and some stationary* $T \in \mathbb{R}^\lambda_\delta$.

Proof. The proof of this Lemma can be worked out along the same lines of the proof of Theorem 2.16 in the next subsection. Thus we refer the reader to that proof. □

By Woodin's equivalence above and Lemma 2.9 we get the following:

Theorem 2.12. *Woodin [18, Theorem 2.53]*

Assume V *is a model of* ZFC+ there are class many Woodin cardinals, *and* $\lambda = \nu^+$ *is a successor cardinal in* V.

Then the following are equivalent for any partial order $P \in V$*:*

1. S^{λ}_{P} *is stationary.*

2. $\mathsf{FA}_\nu(P)$ *holds.*

3. *There is a locally complete embedding of P into $\mathbb{R}^\lambda_\delta$ for some Woodin cardinal $\delta > |P|$.*

SSP denote the class of posets which preserve stationary subsets of ω_1. Martin's maximum MM asserts that $\mathsf{FA}_{\aleph_1}(P)$ holds for all $P \in \mathsf{SSP}$.

The following sums up the current state of affair regarding the classes Γ_λ for $\lambda \leq \aleph_2$.

Theorem 2.13. *Assume there are class many Woodin cardinals. Then:*

1. *$\Gamma_{\aleph_1}$ is the class of all posets which regularly embeds into some $\mathbb{R}^{\aleph_1}_\delta$.*

2. *$\mathbb{R}^{\aleph_2}_\delta \in \mathsf{SSP}$ for any Woodin cardinal δ.*

3. *MM holds if and only if SSP is the class of all posets which regularly embeds into $\mathbb{R}^{\aleph_2}_\delta$ for some Woodin cardinal δ. (Foreman, Magidor, Shelah [5]).*

Proof. We sketch a proof.

1 Trivial by Theorem 2.12.

2 Let $S \in V$ be a stationary subset of ω_1, G be V-generic for $\mathbb{R}^{\aleph_2}_\delta$ and $\dot{C}$ be a $\mathbb{R}^{\aleph_2}_\delta$-name for a club subset of ω_1. Then $\sigma_G(\dot{C}) \in (H_{\omega_2})^{V[G]} = V_\delta[G] = (H_{\omega_2})^{M^G}$. In particular there is some $f : P(V_\alpha) \to P(\omega_1)$ in V_δ such that $[f]_G = \sigma_G(\dot{C})$. By Theorem 2.10.3 the set of $M \prec V_\alpha$ such that $f(M)$ is a club subset of ω_1 in V belongs to G. Thus $f(M) \cap S$ is non empty for all such M, in particular $M^G \models [f]_G \cap j_G(S) \neq \emptyset$. Now, since $j_G(\omega_1) = \omega_1$, we have that $j_G(S) = S$. The conclusion follows.

3 $\aleph_2$ is a a successor cardinal. For this reason, if MM holds, we can use the equivalence given by Theorem 2.12 to get that any $P \in \mathsf{SSP}$ regularly embeds into some $\mathbb{R}^{\aleph_2}_\delta$. We can then use 2 to argue that if P regularly embeds into some $\mathbb{R}^{\aleph_2}_\delta$ with δ a Woodin cardinal, then $P \in \mathsf{SSP}$. $\square$

2.5 Preliminaries V: MM^{++}

The ordinary proof of MM actually gives more information than what is captured by Theorem 2.13.3: the latter asserts that any stationary set preserving poset $\mathbb{P}$ can be completely embedded into $\mathbb{R}^{\aleph_2}_{\delta} \restriction S^{\aleph_2}_{\mathbb{P}}$ for any Woodin cardinal $\delta > |\mathbb{P}|$ via some complete embedding i. However MM doesn't give much information on the nature of the complete embedding i. On the other hand the standard model of MM provided by Foreman, Shelah and Magidor's consistency proof actually show that for any stationary set preserving poset $\mathbb{P}$ and any Woodin cardinal $\delta > |\mathbb{P}|$ we can get a complete embedding $i : \mathbb{P} \to \mathbb{B}(\mathbb{R}^{\aleph_2}_{\delta} \restriction T)$ with a "nice" quotient forcing $(\mathbb{R}^{\aleph_2}_{\delta} \restriction T)/i[\mathbb{P}]$. For this reason we introduce the following well known variation of Martin's maximum:

Definition 2.14. MM^{++} holds if $T_{\mathbb{P}}$ is stationary for all $\mathbb{P} \in \mathsf{SSP}$, where $M \in T_{\mathbb{P}}$ iff

- $M \prec H_{|\mathbb{P}|^+}$ is in $R_{\aleph_2}$,
- there is an M-generic filter H for $\mathbb{P}$ such that, if $G = \pi_M[H]$, $Q = \pi_M(\mathbb{P})$ and $N = \pi_M[M]$, then $\sigma_G : N^Q \to N[G]$ is an evaluation map such that $\sigma_G(\pi_M(\dot{S}))$ is stationary for all $\dot{S} \in M$ $\mathbb{P}$-name for a stationary subset of ω_1.

The following is a well-known by-product of the ordinary consistency proofs of MM which to my knowledge is seldom explicitly stated:

Theorem 2.15 (Foreman, Magidor, Shelah). *Assume κ is supercompact in V, $f : \kappa \to V_\kappa$ is a Laver function and*

$$\{(P_\alpha, \dot{Q}_\alpha) : \alpha \leq \kappa\}$$

is a revised countable support iteration such that

- $P_\alpha \Vdash \dot{Q}_\alpha$ *is semiproper,*
- $P_{\alpha+1} \Vdash |P_\alpha| = \aleph_1$,
- $\dot{Q}_\alpha = f(\alpha)$ *if* $P_\alpha \Vdash f(\alpha)$ *is semiproper.*

Let G be V-generic for P_κ. Then MM^{++} *holds in $V[G]$.*

Theorem 2.16. *Assume there are class many Woodin cardinals. Then the following are equivalent:*

1. MM^{++} *holds.*

2. *For every Woodin cardinal δ and every stationary set preserving poset $\mathbb{P} \in V_\delta$ there is a complete embedding $i : \mathbb{P} \to \mathbb{B}$ where $\mathbb{B} = \mathbb{B}(\mathbb{R}_\delta^{\aleph_2} \upharpoonright T)$ for some stationary set $T \in V_\delta$ such that*

$$\Vdash_{\mathbb{P}} \mathbb{B}/i[\mathbb{P}] \text{ is stationary set preserving.}$$

The rest of this section is devoted to the proof of the above theorem.

Proof. We prove both implications as follows:

1 implies 2 We will show that if G is V-generic for $\mathbb{R}_\delta$ with $T_{\mathbb{P}} \in G$ there is in $V[G]$ a V-generic filter H for $\mathbb{P}$ such that $\sigma_H(\dot{S})$ is stationary in $V[G]$ for all $\mathbb{P}$-names $\dot{S}$ for stationary subsets of ω_1. Assume this is the case and let $\dot{H}$ be a $\mathbb{R}_\delta$-name for H such that $T_{\mathbb{P}}$ force the above property of $\dot{H}$ and $\mathbb{B} = \mathbb{B}(\mathbb{R}_\delta^{\aleph_2} \upharpoonright T_{\mathbb{P}})$. Then it is easy to check that the map

$$\begin{aligned} i : \mathbb{P} &\to \mathbb{B} \\ p &\mapsto \|p \in \dot{H}\|_{\mathbb{B}} \end{aligned}$$

is a complete embedding such that

$$\Vdash_{\mathbb{P}} \mathbb{B}/i[\mathbb{P}] \text{ is stationary set preserving.}$$

To define $\dot{H}$ we proceed as follows: for each $M \in T_{\mathbb{P}}$ let $H_M \in V$ be $\pi_M[M]$-generic for $\pi_M(\mathbb{P})$ and such that $\sigma_{H_M}(\pi_M(\dot{S})) = S_M \in V$ is a stationary subset of ω_1 for all $\mathbb{P}$-name $\dot{S} \in M$ for a stationary subset of ω_1. Thus $[\langle H_M : M \in T_{\mathbb{P}}\rangle]_G$ is V-generic for $\mathbb{P}$. Let $\dot{C}$ be a $\mathbb{R}_\delta$-name for a club subset of ω_1. As in the proof that $\mathbb{R}_\delta$ is stationary set preserving we can argue that $\sigma_G(\dot{C}) = [\langle C_M : M \prec V_\alpha\rangle]_G \in M^G$ is such that $C_M \in V$ is a club subset of ω_1 for some $\alpha < \delta$ and for all $M \prec V_\alpha$. Then

$$\sigma_G(\dot{C}) \cap \sigma_H(\dot{S}) = [\langle C_M \cap S_M : M \in T_{\mathbb{P}}^{V_\alpha}\rangle]_G \neq \emptyset$$

This shows that $[\langle H_M : M \in T_{\mathbb{P}}\rangle]_{\dot{G}}$ is the desired $\mathbb{R}_\delta$-name $\dot{H}$, given that $\dot{G}$ is the canonical $\mathbb{R}_\delta$-name for a V-generic filter for $\mathbb{R}_\delta$.

2 implies 1.

Let $\dot{G}$ be the canonical $\mathbb{R}_\delta$-name for a V-generic filter for $\mathbb{R}_\delta$. Let $T \in \mathbb{R}_\delta$ be a condition such that $\mathbb{P}$ completely embeds into $\mathbb{B} = \mathbb{B}(\mathbb{R}_\delta \upharpoonright T)$ via i and

$$\Vdash_{\mathbb{P}} \mathbb{B}/i[\mathbb{P}] \text{ is stationary set preserving}$$

Let $\dot{G}$ be the canonical name for the $\mathbb{R}_\delta$-generic filter and $\dot{H} = i^{-1}[\dot{G}]$. Then $i(p) = \|p \in \dot{H}\|_{\mathbb{B}}$.

Now notice that $\mathbb{P} \in V_\delta$ and each $\mathbb{P}$-name $\dot{S}$ for a stationary subset of ω_1 is in V_δ since it is given by ω_1-many maximal antichains of $\mathbb{P}$.

Thus if G is V-generic for $\mathbb{R}_\delta$ with $T \in G$, $H = \sigma_G(\dot{H}) \in V_\delta[G]$ is V-generic for $\mathbb{P}$ and is such that $\sigma_H(\dot{S}) \in V_\delta[G]$ is stationary in $V[G]$ for all names $\dot{S} \in V^{\mathbb{P}}$ for stationary subsets of ω_1. Since $V_\delta[G] = (H_{\omega_2})^{M^G}$, $H \in M^G$, so $H = [f]_G$ for some $f : P(V_\alpha) \to P(\mathbb{P})$. It is possible to check that for some $\alpha < \delta$

$$\begin{aligned} S = \{&M \prec V_\alpha : f(M) = H_M \text{ is a } \pi_M[M]\text{-generic filter for } \pi_M(\mathbb{P}) \text{ such that} \\ &\sigma_{H_M}(\pi_M(\dot{S})) \text{ is stationary for all names } \dot{S} \in V^{\mathbb{P}} \cap M \\ &\text{for stationary subsets of } \omega_1\} \in G \end{aligned}$$

In particular $S \leq T_{\mathbb{P}}$ is stationary and we are done. □

3 Absoluteness of the theory of $H_{\aleph_2}$ in models of MM^{++}

In this section we prove Theorem 1.4. We leave to the reader to convert it into a proof of Theorem 1.5

Theorem 3.1. *Assume* MM^{++} *holds in V and there are class many Woodin cardinals. Then the* Π_2*-theory of* $H_{\aleph_2}$ *with parameters cannot be changed by stationary set preserving forcings which preserve* BMM.

Proof. Assume V models MM^{++} and let $P \in M$ be such that V^P models BMM.

Let δ be a Woodin cardinal larger than $|P|$. By Theorem 2.16 there is a complete embedding $i : P \to Q = \mathbb{R}_\delta \upharpoonright T_P$ for some stationary set $T_P \in V_\delta$ such that

$$\Vdash_P Q/i[P] \text{ is stationary set preserving.}$$

Now let G be V-generic for Q and $H = i^{-1}[G]$ be V generic for P. Then $V \subset V[H] \subset V[G]$ and $V[G]$ is a generic extension of $V[H]$ by a forcing which is stationary set preserving in $V[H]$. Moreover by Woodin's theorem on stationary tower forcing 2.10, we have that $H_{\aleph_2}^V \prec H_{\aleph_2}^{V[G]}$.

We show that

$$H_{\aleph_2}^V \prec_{\Sigma_2} H_{\aleph_2}^{V[H]}.$$

This will prove the Theorem, modulo standard forcing arguments.

We have to prove the following for any Σ_0-formula $\phi(x, y, z)$:

1. If
$$H_{\aleph_2}^V \models \exists y \forall x \phi(x, y, p)$$
for some $p \in H_{\aleph_2}^V$, then also
$$H_{\aleph_2}^{V[H]} \models \exists y \forall x \phi(x, y, p).$$

2. If
$$H_{\aleph_2}^V \models \forall y \exists x \phi(x, y, p)$$
for some $p \in H_{\aleph_2}^V$, then also
$$H_{\aleph_2}^{V[H]} \models \forall y \exists x \phi(x, y, p).$$

To prove 1 we note that for some $q \in H_{\aleph_2}^V$ we have that

$$H_{\aleph_2}^V \models \forall x \phi(x, q, p).$$

Then, since

$$H_{\aleph_2}^V \prec H_{\aleph_2}^{V[G]},$$

we have that

$$H_{\aleph_2}^{V[G]} \models \forall x \phi(x, q, p).$$

In particular, since $q, p \in H_{\aleph_2}^{V[H]}$ and $H_{\aleph_2}^{V[H]}$ is a transitive substructure of $H_{\aleph_2}^{V[G]}$, we get that

$$H_{\aleph_2}^{V[H]} \models \forall x \phi(x, q, p)$$

as well. The conclusion now follows.

To prove 2 we note that, since

$$H_{\aleph_2}^V \prec H_{\aleph_2}^{V[G]},$$

we have that

$$H_{\aleph_2}^{V[G]} \models \forall y \exists x \phi(x, y, p).$$

In particular we have that for any $q \in H_{\aleph_2}^{V[H]}$ we have that

$$H_{\aleph_2}^{V[G]} \models \exists x \phi(x, q, p).$$

Now, since $V[H]$ models BMM and $V[G]$ is an extension of $V[H]$ by a stationary set preserving forcing, we get that

$$H_{\aleph_2}^{V[H]} \prec_{\Sigma_1} H_{\aleph_2}^{V[G]}.$$

In particular we can conclude that

$$H_{\aleph_2}^{V[H]} \models \exists x \phi(x, q, p)$$

for all $q \in H_{\aleph_2}^{V[H]}$, from which the desired conclusion follows.

The proof of the theorem is completed. □

4 Woodin's absoluteness results for $H_{\aleph_1}$

Motivated by the results of the previous section we prove the following theorem:

Theorem 4.1. *Assume there are class many Woodin cardinals. Then the theory of $H_{\aleph_1}$ is invariant with respect to set forcing.*

Proof. We prove by induction on n the following Lemma, of which the Theorem is an immediate consequence:

Lemma 4.2. *Assume V is a model of* ZFC *in which there are class many Woodin cardinals. Let $P \in V$ be a forcing notion.*

Then for all n, $H_{\aleph_1}^V \prec_{\Sigma_n} H_{\aleph_1}^{V^P}$.

Proof. By Cohen's absoluteness Lemma 1.2, we already know that for all models M of ZFC and all forcing $P \in M$

$$H_{\aleph_1}^M \prec_{\Sigma_1} H_{\aleph_1}^{M^P}.$$

Now assume that for all models M of ZFC+*there are class many Woodin cardinals* and all $P \in M$ we have shown that

$$H_{\aleph_1}^M \prec_{\Sigma_n} H_{\aleph_1}^{M^P}.$$

First observe that M^P is still a model of ZFC+*there are class many Woodin cardinals*. Now pick V an arbitrary model of ZFC+*there are class many Woodin cardinals* and $P \in V$ a forcing notion.

Let $\delta \in V$ be a Woodin cardinal in V such that $P \in V_\delta$.

To simplify the argument we assume V is transitive and there is a V-generic filter G for $\mathbb{R}_\delta^{\aleph_1}$ (we leave to the reader to remove these unnecessary assumptions).

Then, since $\mathsf{FA}_{\aleph_0}(P)$ holds in V and $P \in V_\delta$, by Theorem 2.13.1 there is in V a complete embedding $i : P \to \mathbb{R}_\delta^{\aleph_1}$. Let $H = i^{-1}[G]$. Then by our inductive assumptions applied to V (with respect to $V[H]$) and to $V[H]$ (with respect to $V[G]$) we have that:

$$H_{\aleph_1}^V \prec_{\Sigma_n} H_{\aleph_1}^{V[H]} \prec_{\Sigma_n} H_{\aleph_1}^{V[G]}.$$

By Woodin's work on the stationary tower forcing we also know that

$$H_{\aleph_1}^V \prec H_{\aleph_1}^{V[G]}.$$

Now we prove that

$$H_{\aleph_1}^V \prec_{\Sigma_{n+1}} H_{\aleph_1}^{V[H]}.$$

Since this argument holds for any V, P and G, the proof will be completed.

We have to prove the following for any Σ_n-formula $\phi(x,z)$ and any Π_n-formula $\psi(x,z)$:

1. If
$$H_{\aleph_1}^V \models \forall x \phi(x,p)$$
for some $p \in \mathbb{R}^V$, then also
$$H_{\aleph_1}^{V[H]} \models \forall x \phi(x,p).$$

2. If
$$H_{\aleph_1}^V \models \exists x \psi(x,p)$$
for some $p \in \mathbb{R}^V$, then also
$$H_{\aleph_1}^{V[H]} \models \exists x \psi(x,p).$$

To prove 1 we note that, since $H_{\aleph_1}^V \prec H_{\aleph_1}^{V[G]}$, we have that

$$H_{\aleph_1}^{V[G]} \models \forall x \phi(x,p).$$

In particular we have that for any $q \in H_{\aleph_1}^{V[H]}$ we have that $H_{\aleph_1}^{V[G]}$ models that $\phi(q, p)$. Now, since by inductive assumptions

$$H_{\aleph_1}^{V[H]} \prec_{\Sigma_n} H_{\aleph_1}^{V[G]},$$

we get that

$$H_{\aleph_1}^{V[H]} \models \phi(q, p)$$

for all $q \in H_{\aleph_1}^{V[H]}$, from which the desired conclusion follows.

To prove 2 we note that for some $q \in H_{\aleph_1}^{V}$ we have that

$$H_{\aleph_1}^{V} \models \psi(q, p).$$

Then, since by inductive assumptions we have that

$$H_{\aleph_1}^{V} \prec_{\Sigma_n} H_{\aleph_1}^{V[H]},$$

we conclude that

$$H_{\aleph_1}^{V[H]} \models \psi(q, p).$$

The conclusion now follows.

The lemma is now completely proved. $\square$

The Theorem is proved. $\square$

Remark 4.3. Theorem 4.1 has a weaker conclusion than [10, Theorem 3.1.7] where it is shown that in the presence of class many inaccessible limits of Woodin cardinals, the first order theory of $L(P_{\omega_1}\mathrm{Ord})$ is invariant with respect to set forcing. However in Theorem 4.1 we have slightly weakened the large cardinal hypothesis with respect to Woodin's [10, Theorem 3.1.7].

We had to weaken the conclusion of Theorem 4.1 with respect to [10, Theorem 3.1.7] since we cannot replace $H_{\aleph_1}$ with $L(\mathbb{R})$ (or $L(P_{\omega_1}\mathrm{Ord})$) in the proof of the above Lemma. The reason is that any element of $L(\mathbb{R})$ is defined by an arbitrarily large ordinal and a real and the ordinal may be moved by j_G, where $j_G : V \to M^G$ is the ultarpower embedding living in $V[G]$ and induced by G. In particular we have that $j_G \restriction H_{\omega_1}^{V}$ is the identity and witnesses that

$$H_{\aleph_1}^{V} \prec H_{\aleph_1}^{V[G]},$$

but j_G may not witness that

$$L(\mathbb{R})^{V} \prec L(\mathbb{R})^{V[G]},$$

which is what we would need in order to perform the type of argument we performed in the proof of the Lemma.

5 Questions and open problems

5.1 A conjecture on MM^{++} and $\Gamma_{\aleph_2}$-logic.

We conjecture the following:

Conjecture 5.1. *Assume V is a model of* MM^{++}+large cardinals. *Then for every $P \in V$ which preserves* MM^{++}

$$H_{\aleph_2}^{V} \prec H_{\aleph_2}^{V^P},$$

There is a major obstacle in performing the arguments of Lemma 4.2 in combination with the proof of Theorem 3.1 to prove this conjecture.

Assume H is V-generic for P and G is V-generic for $\mathbb{R}_{\delta}^{\aleph_2}$ so that:

- $V \models \mathsf{MM}^{++}$,
- $V[H] \models \mathsf{MM}^{++}$,
- $V[G]$ is an extension of V and of $V[H]$ by a stationary set preserving forcing,
- $H_{\aleph_2}^{V} \prec_{\Sigma_1} H_{\aleph_2}^{V[H]} \prec_{\Sigma_1} H_{\aleph_2}^{V[G]}$,
- $H_{\aleph_2}^{V} \prec H_{\aleph_2}^{V[G]}$,

From these data following the proof of Theorem 3.1 we can infer $H_{\aleph_2}^{V} \prec_{\Sigma_2} H_{\aleph_2}^{V[H]}$, but we cannot infer $H_{\aleph_2}^{V[H]} \prec_{\Sigma_2} H_{\aleph_2}^{V[G]}$ (which is what allows us to perform the next step in Lemma 4.2) because we cannot prove that $V[G]$ is a model of BMM (and we do not expect this to be the case).

Thus some new idea is required to prove (or disprove) this conjecture.

5.2 What is the relation between MM^{++} and axiom $(*)$?

It is well known that Woodin's $(*)$-axiom is not compatible with the existence of a well order of $P(\omega_1)$ definable in $H_{\aleph_2}$ without parameters. On the other hand Larson has shown that $\mathsf{MM}^{+\omega}$ is compatible with the existence of a well-order of $P(\omega_1)$ definable in H_{ω_2} without parameters [8]. If we inspect Larson's result, we see that Larson's well order is neither Π_2-definable nor Σ_2-definable over H_{ω_2}. Thus Theorem 1.5 does not prove that Larson's well-order can be defined in all models of MM^{++}. Whether MM^{++} can imply or deny axiom $(*)$ is an interesting open problem. Larson's result already shows that any version of $\mathsf{MM}^{+\alpha}$ strictly weaker than MM^{++} neither denies nor implies axiom $(*)$. However for our absoluteness results it seems to be crucial that the ground model satisfy MM^{++}.

Question 5.2. *Does* MM^{++}+large cardinals *denies or implies Woodin's* $(*)$*-axiom?*

5.3 A conjecture on $\Gamma_{\aleph_3}$-logic

The results of this paper suggest the following definitions.

Let Γ be a class of partial orders defined by some parameter λ which is a regular cardinal definable in some theory T extending ZFC. Natural examples of such classes Γ for $\lambda = \aleph_2$ are the family of stationary set preserving posets, semiproper posets, proper posets, CCC posets. . . .

$\phi(\Gamma, \lambda)$ asserts that

> For all $\mathbb{P} \in \Gamma$ and all Woodin cardinal $\delta > |\mathbb{P}|$ there is a complete embedding $i : \mathbb{P} \to \mathbb{R}^{\lambda}_{\delta} \restriction S$ for some stationary set $S \in V_\delta$ such that
>
> $$\Vdash_{\mathbb{P}} (\mathbb{R}^{\lambda}_{\delta} \restriction S)/i[\mathbb{P}] \in (\Gamma)^{V^{\mathbb{P}}}$$

Notice that $\phi(\Gamma, \lambda)$ entails that $\Gamma = \Gamma_\lambda$ by Woodin's theorem 2.11.

Definition 5.3. A definable class of posets Γ is maximal for λ with respect to the theory T if T models the following:

1. $\Gamma_\lambda \subseteq \Gamma$.
2. $Con(T) \to Con(T + \phi(\Gamma, \lambda))$.
3. $\mathbb{R}^{\lambda}_{\delta} \in \Gamma$ for all Woodin cardinals $\delta > \lambda$.
4. If $i : \mathbb{P} \to \mathbb{Q}$ is a locally complete embedding and $\mathbb{Q} \in \Gamma$, then $\mathbb{P} \in \Gamma$ as well.
5. If for some definable class Γ', $Con(T + \Gamma' \setminus \Gamma \neq \emptyset)$ then $\Gamma' = \Gamma_\lambda$ is not consistent with T.

Remark 5.4. Notice that if $\delta_1 < \delta_2$ are Woodin cardinals, then $\mathbb{R}^{\lambda}_{\delta_1} \restriction T$ completely embeds into $\mathbb{R}^{\lambda}_{\delta_2} \restriction T$ for all regular cardinals $\lambda < \delta_1$ and all stationary sets $T \in P(R_\lambda) \cap V_{\delta_1}$ (see [10, Exercise 2.7.15] and [10, Lemma 2.7.14, Lemma 2.7.16]).

Remark 5.5. The following holds:

1. The class of all posets is maximal for $\aleph_1$ relative to ZFC+ *there are class many Woodin cardinals*.

2. SSP is maximal for $\aleph_2$ relative to ZFC+ *there are class many supercompact cardinals*.

Conjecture 5.6. *There is a class $\Gamma_{\aleph_3}$ which is maximal with respect to the theory* ZFC + MM^{++}+ large cardinals.

Notice that if the above conjecture stands, one should expect to be able to prove the analogue of Theorem 3.1 for $H_{\aleph_3}$.

If the above approach is successful at $\aleph_3$, is there a cardinal for which it cannot work? I.e.:

Question 5.7. *What about maximal classes Γ_λ for larger cardinals λ?*

5.4 What about the effects of MM^{++} on the theory of $L(P_{\omega_2}\mathrm{Ord})$?

Can the methods presented in this paper be of some use in the study of $L(P_{\omega_2}\mathrm{Ord})$ and not just of $H_{\aleph_2}$?

Question 5.8. *Can* MM^{++}+large cardinals *decide in* SSP-*logic the theory of $L(P_{\omega_2}\mathrm{Ord})$?*

While we can effectively compute many of the consequences MM^{++} has on the theory of $H_{\aleph_2}$, this is not the case for $L(P_{\omega_2}\mathrm{Ord})$, for example: by [10, Remark 1.1.28] ZFC fails in $L(P_\kappa\mathrm{Ord})$ for all cardinals κ if there are κ^+-many measurable cardinals in V.

Question 5.9. *Assume* MM^{++} *holds. What is the least ordinal λ for which*

$$H_\lambda^{L(P_{\omega_2}\mathrm{Ord})} \not\models \mathsf{ZFC}?$$

It is not hard to see that in models of MM^{++} the several examples of definable (with parameter in H_{ω_2}) well-orders of $P(\omega_1)$ provided by results of Aspero, Caicedo, Larson, Moore, Todorčević, Veličković, Woodin and others show that λ is larger than $\aleph_2$ and is at most the $\omega_2 + 1$-th measurable cardinal of V.

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